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A MODEL SYSTEM FOR THE BEHAVIOR OF TWO NON-LINEARLY COUPLED OSCILLATORS

A. Maccari

Technical Institute "G. Cardano", P.za della Resistenza 1, 00015 *Monterotondo RM*, *Italy*

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This paper presents a systematic analysis approach for the study of two non-linearly coupled oscillators, with incommensurable fundamental frequencies. The asymptotic perturbation method is used to analyze a bifurcation problem of codimension two. It is found that the amplitude modulation equations are equal to normal forms equations available in the literature. Approximate analytic solutions for generic quadratic and cubic non-linearities can be constructed. The results obtained from a computer simulation based on a fifth order Runge–Kutta–Fehlberg scheme confirm the validity of the asymptotic perturbation method. The method is illustrated by applying it to a two-rod system subjected to aerodynamic excitation.

1. INTRODUCTION

The study of non-linear oscillations has occupied researchers for quite a long period of time. Various perturbation methods such as averaging, harmonic balance, Lindstedt–Poincaré and multiple scales have been used to construct approximate solutions for weakly non-linear oscillations. The method of averaging has been for a long time an important method of analysis of non-linear systems [1–3]. One must stress that such analysis is usually carried out only to the first order approximation since a second order calculation does not change the qualitative behavior of the solution. However, there are situations in which second order averaging is necessary to obtain correct qualitative information [4].

The multiple scales method was introduced by Nayfeh and has been applied to non-linear oscillations with subharmonic and superharmonic resonances and to modal interactions [5–8]. This method has proved to be a powerful tool in determining small amplitude periodic solutions and their stability.

The harmonic balance method has also been widely used to study the stable and unstable periodic solutions of non-linear oscillators. A satisfying expression for the periodic solution is obtained by using a sufficiently large number of harmonics which often leads to messy algebraic manipulations [9–11]. This problem has been resolved by the use of computer symbolic programs to construct the resulting non-linear algebraic equations [12].

The Lindstedt–Poincaré method can be used in order to find steady-state bifurcating solutions, but stability analysis is usually more complex [8, 13].

Perturbation methods can obtain approximate analytical solutions, and investigate the existence, uniqueness, stability of the solutions and their dependence on parametric values.

On the other hand, local bifurcation theory can be used to reduce a multidimensional dynamical system to a lower dimensional equivalent system and to determine the qualitative behavior of the system. By carrying out center manifold reduction, one must describe the manifold on which the steady-state dynamics takes place and the solution of

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a functional equation is required. In practice, the functional equation is solved only up to a few terms of the Taylor series, sufficient to determine the dynamics. Subsequently, a normal form theory is used that converts an ordinary differential equation to a simpler equation by using a sequence of appropriate changes of variables. The method using normal forms has been only recently extensively used in applied mathematics [4, 14, 15].

However, to obtain accurate quantitative information about the original system one must calculate the explicit expressions of the coefficients of the reduced system in terms of the coefficients of the original system, because these expressions are not available in literature. A large computational effort is required (particularly in the case of high codimension bifurcations), because the procedure must be repeated for each specific problem.

Recently a new asymptotic perturbation (AP) method has been proposed [16–18]. It is based on large temporal rescalings and balancing of harmonic terms with a simple iteration. In a certain sense the AP method can be considered as an attempt to link the most useful characteristics of harmonic balance and multiple scale methods.

In this paper the AP method is used to study a classical problem in non-linear mechanics, i.e. the mutual interaction of two weakly dissipative oscillators. In particular, the characteristics of a system formed by two unidimensional oscillators, which have different fundamental frequencies and are coupled by means of non-linear terms are studied. The relevant system of differential equations is

$$\ddot{X}(t) + \omega_1^2 X(t) = a_1 \dot{X}(t) + F_1(X(t), \dot{X}(t), Y(t), \dot{Y}(t)) \tag{1a}$$

$$\ddot{Y}(t) + \omega_2^2 Y(t) = b_1 \dot{Y}(t) + F_2(X(t), \dot{X}(t), Y(t), \dot{Y}(t)) \tag{1b}$$

where dot denotes differentiation with respect to the time, ω_1 and ω_2 are the uncoupled natural frequencies, a_1, b_1 are small dissipative terms and the functions $F_1(X(t), \dot{X}(t), Y(t), \dot{Y}(t))$ and $F_2(X(t), \dot{X}(t), Y(t), \dot{Y}(t))$ represent the non-linear coupling. If one considers only small oscillations ($X(t), Y(t) \ll 1$) and if F_1 and F_2 are analytic, they can be expanded in power series retaining only quadratic and cubic terms to give

$$\begin{aligned}\ddot{X}(t) + \omega_1^2 X(t) = {} & a_1 \dot{X}(t) + a_2 X^2(t) + a_3 X(t)\dot{X}(t) + a_4 \dot{X}^2(t) \\ & + a_5 X(t)Y(t) + a_6 \dot{X}(t)Y(t) + a_7 Y^2(t) + a_8 Y(t)\dot{Y}(t) + a_9 \dot{Y}^2(t) \\ & + a_{10} X(t)\dot{Y}(t) + a_{11}\dot{X}(t)\dot{Y}(t) + a_{12} X^3(t) + a_{13} X^2(t)\dot{X}(t) \\ & + a_{14} X(t)\dot{X}^2(t) + a_{15}\dot{X}^3(t) + a_{16} X(t)Y^2(t) + a_{17}\dot{X}(t)Y^2(t) \\ & + a_{18} X(t)\dot{Y}^2(t) + a_{19}\dot{X}(t)\dot{Y}^2(t) + a_{20} X(t)Y(t)\dot{Y}(t),\end{aligned} \tag{2a}$$

$$\begin{aligned}\ddot{Y}(t) + \omega_2^2 Y(t) = {} & b_1 \dot{Y}(t) + b_2 X^2(t) + b_3 X(t)\dot{X}(t) + b_4 \dot{X}^2(t) \\ & + b_5 X(t)Y(t) + b_6 \dot{X}(t)Y(t) + b_7 Y^2(t) + b_8 Y(t)\dot{Y}(t) + b_9 \dot{Y}^2(t) \\ & + b_{10} X(t)\dot{Y}(t) + b_{11}\dot{X}(t)\dot{Y}(t) + b_{12} Y^3(t) + b_{13} Y^2(t)\dot{Y}(t) \\ & + b_{14} Y(t)\dot{Y}^2(t) + b_{15}\dot{Y}^3(t) + b_{16} X^2(t)Y(t) + b_{17} X^2(t)\dot{Y}(t) \\ & + b_{18}\dot{X}^2(t)Y(t) + b_{19}\dot{X}^2(t)\dot{Y}(t) + b_{20} Y(t)X(t)\dot{X}(t).\end{aligned} \tag{2b}$$

Some cubic terms and all the powers superior to the third have not been included, in the expression of the non-linear coupling, because they would involve a negligible contribution, at least in the first approximation, as we will demonstrate in the following. Note that the coefficients a_i, b_i, with $i = 2, \ldots, 20$, are of order 1.

In section 2, it is demonstrated that the behavior of the very general system (2a)–(2b), with 39 arbitrary parameters (by a suitable variable change we can set $\omega_1 = a_2 = b_7 = 1$), can be represented by a model and universal system, which contains only five arbitrary parameters. These parameters can be expressed in closed form in terms of the coefficients of the original system. Moreover, we do not need to identify the center manifold and to express the Jacobian matrix at the critical state in Jordan form.

In section 3, an approximate solution is obtained, and the conditions for the existence of stable oscillations derived. The model system obtained describes a bifurcation of codimension two, i.e. a non-resonant double Hopf bifurcation.

The approximate solution is then compared with the numerical solution obtained by means of the Runge–Kutta–Fehlberg method.

In section 4, the procedure is applied to a mechanical system composed of two rods under aerodynamic excitation and determines its post-critical behavior.

In the last section, the most important results are summarized and some possible extensions and generalizations indicated.

2. THE FIRST ORDER APPROXIMATE SOLUTION

The AP method derives from a similar method used for the partial differential equations [19–21]. The following temporal rescaling is introduced

$$\tau = \varepsilon^q t, \tag{3}$$

where q is a positive number, which will be fixed afterwards, because it establishes to what extent one can push the temporal asymptotic limit, to permit the non-linear effects to become consistent and not negligible. If $t \to \infty$, then $\varepsilon \to 0$, and τ assumes a finite value. The linear dissipative coefficients a_1, b_1 are supposed of order ε^2 and then the differential equations (2a)–(2b) can be written as

$$\begin{aligned}\ddot{X}(t) + \omega_1^2 X(t) = {} & \varepsilon^2 a_1 \dot{X}(t) + a_2 X^2(t) + a_3 X(t)\dot{X}(t) + a_4 \dot{X}^2(t) \\ & + a_5 X(t) Y(t) + a_6 \dot{X}(t) Y(t) + a_7 Y^2(t) + a_8 Y(t)\dot{Y}(t) + a_9 \dot{Y}^2(t) \\ & + a_{10} X(t)\dot{Y}(t) + a_{11}\dot{X}(t)\dot{Y}(t) + a_{12} X^3(t) + a_{13} X^2(t)\dot{X}(t) \\ & + a_{14} X(t)\dot{X}^2(t) + a_{15}\dot{X}^3(t) + a_{16} X(t) Y^2(t) + a_{17}\dot{X}(t) Y^2(t) \\ & + a_{18} X(t)\dot{Y}^2(t) + a_{19}\dot{X}(t)\dot{Y}^2(t) + a_{20} X(t) Y(t)\dot{Y}(t),\end{aligned} \tag{4a}$$

$$\begin{aligned}\ddot{Y}(t) + \omega_2^2 Y(t) = {} & \varepsilon^2 b_1 \dot{Y}(t) + b_2 X^2(t) + b_3 X(t)\dot{X}(t) + b_4 \dot{X}^2(t) \\ & + b_5 X(t) Y(t) + b_6 \dot{X}(t) Y(t) + b_7 Y^2(t) + b_8 Y(t)\dot{Y}(t) + b_9 \dot{Y}^2(t) \\ & + b_{10} X(t)\dot{Y}(t) + b_{11}\dot{X}(t)\dot{Y}(t) + b_{12} Y^3(t) + b_{13} Y^2(t)\dot{Y}(t) \\ & + b_{14} Y(t)\dot{Y}^2(t) + b_{15}\dot{Y}^3(t) + b_{16} X^2(t) Y(t) + b_{17} X^2(t)\dot{Y}(t) \\ & + b_{18}\dot{X}^2(t) Y(t) + b_{19}\dot{X}^2(t)\dot{Y}(t) + b_{20} Y(t) X(t)\dot{X}(t).\end{aligned} \tag{4b}$$

The requested solution $(X(t), Y(t))$ can be expressed by means of a power series in the expansion parameter ε. One can formally write

$$X(t) = \sum_{n_1, n_2 = -\infty}^{+\infty} \varepsilon^{\gamma_{n_1 n_2}} \psi_{n_1 n_2}(\tau, \varepsilon) \exp(-\mathrm{i}(n_1\omega_1 + n_2\omega_2)t), \tag{5a}$$

$$Y(t) = \sum_{n_1, n_2 = -\infty}^{+\infty} \varepsilon^{\tilde{\gamma}_{n_1 n_2}} \phi_{n_1 n_2}(\tau, \varepsilon) \exp(-\mathrm{i}(n_1\omega_1 + n_2\omega_2)t), \tag{5b}$$

with $\gamma_{n_1n_2} = |n_1| + |n_2|$ for $n_1 \neq 0$ and $(n_1, n_2) \neq (0, 0)$ (and then n_2 may be 0), $\gamma_{0n_2} = 2 + |n_2|$ for $n_2 \neq 0$, $\tilde{\gamma}_{n_1n_2} = |n_1| + |n_2|$ for $n_2 \neq 0$ and for $(n_1, n_2) \neq (0, 0)$ (and then n_1 may be 0), $\tilde{\gamma}_{n_10} = 2 + |n_1|$ for $n_1 \neq 0$, and $\gamma_{00} = \tilde{\gamma}_{00} = r$ a non-negative number, which will be fixed later on. Note that $\psi_{n_1n_2}(\tau, \varepsilon) = \psi^*_{-n_1n_2}(\tau, \varepsilon)$ and $\phi_{n_1n_2}(\tau, \varepsilon) = \phi^*_{-n_1n_2}(\tau, \varepsilon)$, because $X(t)$ and $Y(t)$ are real. The functions $\psi_{n_1n_2}(\tau, \varepsilon)$, $\phi_{n_1n_2}(\tau, \varepsilon)$ depend on the parameter ε and suppose that their limit for $\varepsilon \to 0$ exists and is finite and moreover they can be expanded in power series of ε, i.e.

$$\psi_{n_1n_2}(\tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \psi^{(i)}_{n_1n_2}(\tau), \qquad \phi_{n_1n_2}(\tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \phi^{(i)}_{n_1n_2}(\tau). \tag{5c}$$

In the following for simplicity the abbreviations $\psi^{(0)}_{n_1n_2} = \psi_{n_1n_2}$, $\phi^{(0)}_{n_1n_2} = \phi_{n_1n_2}$ is used.

Note that the variable change (3) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_{n_1n_2}\exp(-\mathrm{i}(n_1\omega_1 + n_2\omega_2)t))$$
$$= \left(-\mathrm{i}(n_1\omega_1 + n_2\omega_2)\psi_{n_1n_2} + \varepsilon^q \frac{\mathrm{d}\psi_{n_1n_2}}{\mathrm{d}\tau}\right)\exp(-\mathrm{i}(n_1\omega_1 + n_2\omega_2)t). \tag{5d}$$

The parameter ε represents the expansion constant of the method and it can be considered of arbitrary value, provided that it is sufficiently small. Its comparison in the temporal rescaling (3) permits one to determine the asymptotic behavior of the solution, when the non-linear effects can supply a non-negligible contribution.

The Fourier expansion (5a)–(5b) can be written more explicitly

$$X(t) = \varepsilon^r\psi_{00} + (\varepsilon\psi_{10}\exp(-\mathrm{i}\omega_1 t) + \varepsilon^2\psi_{20}\exp(-2\mathrm{i}\omega_1 t)$$
$$+ \varepsilon^2\psi_{11}\exp(-\mathrm{i}(\omega_1 + \omega_2)t) + \varepsilon^2\psi_{1-1}\exp(-\mathrm{i}(\omega_1 - \omega_2)t) + c.c.) + o(\varepsilon^3), \tag{6a}$$
$$Y(t) = \varepsilon^r\phi_{00} + (\varepsilon\phi_{01}\exp(-\mathrm{i}\omega_2 t) + \varepsilon^2\phi_{02}\exp(-2\mathrm{i}\omega_2 t)$$
$$+ \varepsilon^2\phi_{11}\exp(-\mathrm{i}(\omega_1 + \omega_2)t) + \varepsilon^2\phi_{-11}\exp(\mathrm{i}(\omega_1 - \omega_2)t) + c.c.) + o(\varepsilon^3). \tag{6b}$$

The solution is then a Fourier expansion in which the coefficients are power series of a small parameter (ε) and vary slowly in time. The lowest order terms correspond to the harmonic solution of the linear problem. Evolution equations for the amplitudes of the harmonic terms are then derived by substituting the expression of the solution into the original equations and projecting onto each Fourier component.

After inserting this expansion in the complete equation (4a)–(4b), one obtains some equations, for every harmonic and for a fixed order of approximation, which are right for the purpose to determine the amplitudes of the various harmonic terms.

Developing the calculations one finally arrives at the equations for the coefficients $\psi_{n_1n_2}$, $\phi_{n_1n_2}$ or better for their limit when $\varepsilon \to 0$, which exhibit a universal character, because they are model equations for the very large class of equations (1a)–(1b).

Only $\psi_{10} = \Psi$ and $\phi_{01} = \Phi$ appear in our equations, because every $\psi_{n_1n_2}$ and $\phi_{n_1n_2}$ can be expressed by means of them. Considering equation (4a) for $n_1 = 1$, $n_2 = 0$, gives

$$\frac{\mathrm{d}\psi_{10}}{\mathrm{d}\tau}\varepsilon^{1+q} = \varepsilon^3\frac{a_1}{2}\psi_{10} + \mathrm{i}\frac{a_2}{\omega_1}(\varepsilon^{2+r}\psi_{00}\psi_{10} + \varepsilon^3\psi_{20}\psi_{-10}) + \frac{a_3}{2}(\varepsilon^{2+r}\psi_{00}\psi_{10} + \varepsilon^3\psi_{20}\psi_{-10})$$
$$+ 2\mathrm{i}\omega_1 a_4\varepsilon^3\psi_{20}\psi_{-10} + \frac{\mathrm{i}}{2\omega_1}a_5\varepsilon^{2+r}\phi_{00}\psi_{10} + \frac{a_6}{2}\varepsilon^{2+r}\phi_{00}\psi_{10}$$

$$+\frac{1}{2\omega_1}(\mathrm{i}a_5-\omega_2a_{10}+\mathrm{i}a_{11}\omega_2(\omega_2+\omega_1)+a_6(\omega_1+\omega_2))\varepsilon^3\psi_{11}\phi_{0-1}$$

$$+\frac{1}{2\omega_1}(\mathrm{i}a_5+\omega_2a_{10}+\mathrm{i}a_{11}\omega_2(\omega_2-\omega_1)+a_6(\omega_1-\omega_2))\varepsilon^3\psi_{1-1}\phi_{01}$$

$$+\frac{1}{2\omega_1}(2\mathrm{i}a_7+\omega_1a_8+2\mathrm{i}a_9\omega_2(\omega_2+\omega_1))\varepsilon^3\phi_{11}\phi_{0-1}$$

$$+\frac{1}{2\omega_1}(2\mathrm{i}a_7+\omega_1a_8+2\mathrm{i}a_9\omega_2(\omega_2-\omega_1))\varepsilon^3\phi_{1-1}\phi_{01}$$

$$+\frac{\mathrm{i}}{2\omega_1}(3a_{12}-\mathrm{i}\omega_1a_{13}+\omega_1^2a_{14}-3\mathrm{i}\omega_1^3a_{15})\varepsilon^3|\psi_{10}|^2\psi_{10}$$

$$+\frac{\mathrm{i}}{2\omega_1}(2a_{16}-2\mathrm{i}\omega_1a_{17}+2\omega_2^2a_{18}-2\mathrm{i}\omega_1\omega_2^2a_{19})\varepsilon^3|\phi_{10}|^2\psi_{10}$$

$$+o(\varepsilon^5,\varepsilon^{1+2q},\varepsilon^{4+r}). \tag{7}$$

The presence in (4a)–(b) of the other cubic terms and of powers superior to the third is irrelevant, because their contribution is of ε^5 and then negligible at the first order of approximation. For $n_1=0$, $n_2=0$

$$\varepsilon^r\psi_{00}=\varepsilon^2(A_{00}|\psi_{10}|^2+\tilde{A}_{00}|\varphi_{10}|^2)+o(\varepsilon^4), \tag{8a}$$

$$A_{00}=2\left(\frac{a_2}{\omega_1^2}+a_4\right),\qquad \tilde{A}_{00}=\frac{2}{\omega_1^2}(a_7+a_9\omega_2^2). \tag{8b}$$

A correct balance of linear and non-linear terms demands that $r=2$.

For $n_1=2$, $n_2=0$, we obtain

$$\varepsilon^2\psi_{20}=\varepsilon^2(A_{20}+\mathrm{i}\tilde{A}_{20})\psi_{10}^2+o(\varepsilon^4), \tag{9a}$$

$$A_{20}=\frac{a_4}{3}-\frac{a_2}{3\omega_1^2},\qquad \tilde{A}_{20}=\frac{a_3}{3\omega_1}, \tag{9b}$$

for $n_1=1$, $n_2=1$

$$\varepsilon^2\psi_{11}=\varepsilon^2(A_{11}+\mathrm{i}\tilde{A}_{11})\psi_{10}\phi_{01}+o(\varepsilon^4), \tag{10a}$$

$$A_{11}=\frac{\omega_1\omega_2a_{11}-a_5}{\omega_2(\omega_2+2\omega_1)},\qquad \tilde{A}_{11}=\frac{\omega_1a_6+\omega_2a_{10}}{\omega_2(\omega_2+2\omega_1)}, \tag{10b}$$

and finally for $n_1=1$, $n_2=-1$

$$\varepsilon^2\psi_{1-1}=\varepsilon^2(A_{1-1}+\mathrm{i}\tilde{A}_{1-1})\psi_{10}\phi_{0-1}+o(\varepsilon^4), \tag{11a}$$

$$A_{1-1}=\frac{a_5+\omega_1\omega_2a_{11}}{\omega_2(2\omega_1-\omega_2)},\qquad A_{1-1}=\frac{\omega_2a_{10}-\omega_1a_6}{\omega_2(2\omega_1-\omega_2)}. \tag{11b}$$

A similar procedure can be obviously repeated for the equation (4b).

As a consequence of (7), the choice $q=2$ is necessary for the consistency of the method, because in this way the various predominant terms become of the same magnitude order.

After inserting (8), (9), (10), (11) in (7) and an analogous treatment for the equation (4b), one arrives at the system model

$$\frac{d\Psi}{d\tau} = \alpha_1 \Psi + (\beta_1 + i\tilde{\beta}_1)|\Psi|^2\Psi + (\gamma_1 + i\tilde{\gamma}_1)|\Phi|^2\Psi \tag{12a}$$

$$\frac{d\Phi}{d\tau} = \alpha_2 \Phi + (\beta_2 + i\tilde{\beta}_2)|\Psi|^2\Phi + (\gamma_2 + i\tilde{\gamma}_2)|\Phi|^2\Phi \tag{12b}$$

where

$$\alpha_1 = \frac{a_1}{2}, \qquad \alpha_2 = \frac{b_1}{2}, \tag{13a}$$

$$\beta_1 = \frac{a_6}{2} B_{00} - 2\omega_1 a_4 \tilde{A}_{20} + \frac{a_3}{2} A_{00} + \frac{a_3}{2} A_{20} - \frac{a_2}{\omega_1} \tilde{A}_{20} + \frac{a_{13}}{2} + \frac{3}{2} \omega_1^2 a_{15}, \tag{13b}$$

$$\begin{aligned}
\gamma_1 = {} & \frac{a_3}{2} \tilde{A}_{00} + \frac{a_6}{2} \tilde{B}_{00} \\
& + \frac{1}{2\omega_1} [((\omega_1 + \omega_2) a_6 - \omega_2 a_{10}) A_{11} - (\omega_2(\omega_1 + \omega_2) a_{11} + a_5) \tilde{A}_{11}] \\
& + \frac{1}{2\omega_1} [((\omega_1 - \omega_2) a_6 + \omega_2 a_{10}) A_{1-1} - (\omega_2(\omega_2 - \omega_1) a_{11} + a_5) \tilde{A}_{1-1}] \\
& + \frac{1}{2\omega_1} [\omega_1 a_8 B_{11} - 2\omega_2(\omega_2 + \omega_1) a_9 \tilde{B}_{11} - 2 a_7 \tilde{B}_{11}] \\
& + \frac{1}{2\omega_1} [\omega_1 a_8 B_{1-1} - 2\omega_2(\omega_2 - \omega_1) a_9 \tilde{B}_{1-1} - 2 a_7 \tilde{B}_{1-1}] \\
& + a_{17} + \omega_2^2 a_{19},
\end{aligned} \tag{13c}$$

$$\begin{aligned}
\tilde{\gamma}_1 = {} & \frac{a_2}{\omega_1} \tilde{A}_{00} + \frac{a_5}{2\omega_1} \tilde{B}_{00} \\
& + \frac{1}{2\omega_1} [((\omega_1 + \omega_2) a_6 - \omega_2 a_{10}) \tilde{A}_{11} + (\omega_2(\omega_1 + \omega_2) a_{11} + a_5) A_{11}] \\
& + \frac{1}{2\omega_1} [((\omega_1 - \omega_2) a_6 + \omega_2 a_{10}) \tilde{A}_{1-1} + (\omega_2(\omega_2 - \omega_1) a_{11} + a_5) A_{1-1}] \\
& + \frac{1}{2\omega_1} [\omega_1 a_8 \tilde{B}_{11} + 2\omega_2(\omega_2 + \omega_1) a_9 B_{11} + 2 a_7 B_{11}] \\
& + \frac{1}{2\omega_1} [\omega_1 a_8 \tilde{B}_{1-1} + 2\omega_2(\omega_2 - \omega_1) a_9 B_{1-1} + 2 a_7 B_{1-1}] \\
& + \frac{a_{16}}{\omega_1} + \frac{\omega_2^2 a_{18}}{\omega_1},
\end{aligned} \tag{13d}$$

$$\beta_2 = \frac{b_8}{2} B_{00} + \frac{b_{10}}{2} A_{00}$$
$$+ \frac{1}{2\omega_2}[((\omega_1 + \omega_2)b_{10} - \omega_1 b_6)B_{11} - (\omega_1(\omega_1 + \omega_2)b_{11} + b_5)\tilde{B}_{11}]$$
$$+ \frac{1}{2\omega_2}[((\omega_2 - \omega_1)b_{10} + \omega_1 b_6)B_{1-1} + (\omega_1(\omega_1 - \omega_2)b_{11} + b_5)\tilde{B}_{1-1}]$$
$$+ \frac{1}{2\omega_2}[\omega_2 b_3 A_{11} - 2\omega_1(\omega_2 + \omega_1)b_4\tilde{A}_{11} - 2b_2\tilde{A}_{11}]$$
$$+ \frac{1}{2\omega_2}[\omega_2 b_3 A_{1-1} + 2\omega_1(\omega_1 - \omega_2)b_4\tilde{A}_{1-1} + 2b_2\tilde{A}_{1-1}]$$
$$+ b_{17} + \omega_1^2 b_{19}, \tag{13e}$$

$$\tilde{\beta}_1 = \frac{a_2}{\omega_1} A_{00} + \frac{a_2}{\omega_1} A_{20} + 2\omega_1 a_4 A_{20} + \frac{a_3}{2}\tilde{A}_{20} + \frac{a_5}{2\omega_1} B_{00} + \frac{a_{14}\omega_1}{2} + \frac{3a_{12}}{2\omega_1}, \tag{13f}$$

$$\tilde{\beta}_2 = \frac{b_7}{\omega_2} B_{00} + \frac{b_5}{2\omega_2} A_{00}$$
$$+ \frac{1}{2\omega_2}[((\omega_1 + \omega_2)b_{10} - \omega_1 b_6)\tilde{B}_{11} + (\omega_1(\omega_1 + \omega_2)b_{11} + b_5)B_{11}]$$
$$+ \frac{1}{2\omega_2}[((\omega_1 - \omega_2)b_{10} - \omega_1 b_6)\tilde{B}_{1-1} + (\omega_1(\omega_1 - \omega_2)b_{11} + b_5)B_{1-1}]$$
$$+ \frac{1}{2\omega_2}[\omega_2 b_3\tilde{A}_{11} + 2\omega_1(\omega_2 + \omega_1)b_4 A_{11} + 2b_2 A_{11}]$$
$$+ \frac{1}{2\omega_2}[-\omega_2 b_3\tilde{A}_{1-1} + 2\omega_1(\omega_1 - \omega_2)b_4 A_{1-1} + 2b_2 A_{1-1}]$$
$$+ \frac{b_{16}}{\omega_2} + \frac{\omega_1^2 b_{18}}{\omega_2}, \tag{13g}$$

$$\gamma_2 = \frac{b_{10}}{2}\tilde{A}_{00} - 2\omega_2 b_9\tilde{B}_{02} + \frac{b_8}{2} B_{02} + \frac{b_8}{2}\tilde{B}_{00} - \frac{b_7}{\omega_2}\tilde{B}_{02} + \frac{b_{13}}{2} + \frac{3\omega_2^2 b_{15}}{2}, \tag{13h}$$

$$\tilde{\gamma}_2 = \frac{b_7}{\omega_2}\tilde{B}_{00} + \frac{b_7}{\omega_2} B_{02} + 2\omega_2 b_9 B_{02} + \frac{b_8}{2}\tilde{B}_{02} + \frac{b_5}{2\omega_2}\tilde{A}_{00} + \frac{3b_{12}}{2\omega_2} + \frac{\omega_2 b_{14}}{2}, \tag{13i}$$

with

$$B_{02} = \frac{b_9}{3} - \frac{b_7}{3\omega_2^2}, \qquad \tilde{B}_{02} = \frac{b_8}{3\omega_2} \tag{13j}$$

and B_{00}, $\tilde{B}_{00}$, B_{11}, $\tilde{B}_{11}$, B_{1-1}, $\tilde{B}_{1-1}$, can be obtained from the corresponding A_{ij}, simply exchanging a_j with b_j, and ω_1 with ω_2 only in the denominator. In the first approximation, the behavior of the very general system (4a)–(4b) can be always reconnected to the universal and model system (12a)–(12b), even if in the original system there are non-linear terms with powers superior to the cubic terms, because their magnitude order is at least of order ε^4 and then is negligible. The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and on time-scale $t = O(1/\varepsilon^2)$. If one wishes to study solutions on intervals such that $\tau = O(1/\varepsilon)$ then the higher terms will in general affect the solution and must be included.

3. MULTIPLE BIFURCATIONS AND LIMIT CYCLES

With the substitution

$$\Psi(\tau) = \rho(\tau)\exp(\mathrm{i}\vartheta(\tau)), \qquad \Phi(\tau) = \chi(\tau)\exp(\mathrm{i}\varphi(\tau)), \tag{14}$$

one arrives at the model equations

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \alpha_1\rho + \beta_1\rho^3 + \gamma_1\rho\chi^2, \tag{15a}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \alpha_2\chi + \beta_2\chi\rho^2 + \gamma_2\chi^3, \tag{15b}$$

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \tilde{\beta}_1\rho^2 + \tilde{\gamma}_1\chi^2, \tag{15c}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \tilde{\beta}_2\rho^2 + \tilde{\gamma}_2\chi^2. \tag{15d}$$

The amplitude modulation equations (15a–b), are uncoupled from the phase modulation equations (15c, d) and have three arbitrary parameters, because with a variable change we can always suppose that $\alpha_1 = \pm 1$, $\beta_1 = \pm 1$, $\gamma_2 = \pm 1$. The system (15c)–(15d) has only two arbitrary parameters (we can always set $\tilde{\beta}_1 = \tilde{\beta}_2 = 1$). Equations (15a, b) are the bifurcation equations in standard normal form for a non-resonant double Hopf bifurcation. Their complete classification can be found for instance in reference [4].

Four points of equilibrium must be considered ($\mathrm{d}\rho/\mathrm{d}\tau = 0$, $\mathrm{d}\chi/\mathrm{d}\tau = 0$):

$$\begin{aligned} P_1 &= (\chi_1, \rho_1) = (0, 0), \\ P_2 &= (\chi_2, \rho_2) = \left(\sqrt{-\frac{\alpha_2}{\gamma_2}}, 0\right), \\ P_3 &= (\chi_3, \rho_3) = \left(0, \sqrt{-\frac{\alpha_1}{\beta_1}}\right), \\ P_4 &= (\chi_4, \rho_4) = \left(\sqrt{\frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\gamma_2\beta_1 - \gamma_1\beta_2}}, \sqrt{\frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_2\beta_1 - \gamma_1\beta_2}}\right). \end{aligned} \tag{16}$$

These solutions correspond to one-frequency periodic motion or two-frequency quasi-periodic motions of the original system (2a, b).

Obviously P_2, P_3, P_4 are present only if the arguments of the square roots are non-negative.

The first point is asymptotically stable for $\alpha_1 < 0$ and $\alpha_2 < 0$ (oscillations are continuously damped and the oscillator stops), the second for $\alpha_2 > 0$, $\gamma_2 < 0$ and $\alpha_1 < (\alpha_2\gamma_1)/\gamma_2$ (oscillation with frequency equal to ω_1 is damped and oscillation with frequency ω_2 keeps a constant amplitude), the third point for $\alpha_1 > 0$, $\beta_1 < 0$ and $\alpha_2 < (\alpha_1\beta_2)\beta_1$ (oscillation with frequency equal to ω_2 is damped and oscillation with frequency ω_1 keeps a constant amplitude). The fourth point is asymptotically stable for $\alpha_1\gamma_2 < \alpha_2\gamma_1$, $\alpha_2\beta_1 < \alpha_1\beta_2$ and $\gamma_2(\alpha_2\beta_1 - \alpha_1\beta_2) + \beta_1(\alpha_1\gamma_2 - \alpha_2\gamma_1) > 0$ (both oscillations with frequencies ω_1 and ω_2 keeps a constant amplitude).

The approximate solution valid to order of ε^2 is

$$\begin{aligned} X(t) &= \varepsilon^2(A_{00}\rho^2(t) + \tilde{A}_{00}\chi^2(t)) + 2\varepsilon\rho(t)\cos(-\omega_1 t + \vartheta(t)) \\ &+ 2\varepsilon^2 A_{20}\rho^2(t)\cos(-2\omega_1 t + 2\vartheta(t)) - 2\varepsilon^2\tilde{A}_{20}\rho^2(t)\sin(-2\omega_1 t + 2\vartheta(t)) \\ &+ 2\varepsilon^2 A_{11}\rho(t)\chi(t)\cos(-\omega_1 t - \omega_2 t + \vartheta(t) + \varphi(t)) \\ &- 2\varepsilon^2\tilde{A}_{11}\rho(t)\chi(t)\sin(-\omega_1 t - \omega_2 t + \vartheta(t) + \varphi(t)) \\ &+ 2\varepsilon^2 A_{1-1}\rho(t)\chi(t)\cos(-\omega_1 t + \omega_2 t - \vartheta(t) + \varphi(t)) \\ &- 2\varepsilon^2\tilde{A}_{1-1}\rho(t)\chi(t)\sin(-\omega_1 t + \omega_2 t - \vartheta(t) + \varphi(t)) + o(\varepsilon^3), \end{aligned} \tag{17a}$$

$$\begin{aligned} Y(t) &= \varepsilon^2(B_{00}\rho^2(t) + \tilde{B}_{00}\chi^2(t)) + 2\varepsilon\chi(t)\cos(-\omega_2 t + \varphi(t)) \\ &+ 2\varepsilon^2 B_{02}\chi^2(t)\cos(-2\omega_2 t + 2\varphi(t)) - 2\varepsilon^2\tilde{B}_{02}\chi^2(t)\sin(-2\omega_2 t + 2\varphi(t)) \\ &+ 2\varepsilon^2 B_{11}\rho(t)\chi(t)\cos(-\omega_1 t - \omega_2 t + \vartheta(t) + \varphi(t)) \\ &- 2\varepsilon^2\tilde{B}_{11}\rho(t)\chi(t)\sin(-\omega_1 t - \omega_2 t + \vartheta(t) + \varphi(t)) \\ &+ 2\varepsilon^2 B_{1-1}\rho(t)\chi(t)\cos(-\omega_1 t + \omega_2 t - \vartheta(t) + \varphi(t)) \\ &- 2\varepsilon^2\tilde{B}_{1-1}\rho(t)\chi(t)\sin(-\omega_1 t + \omega_2 t - \vartheta(t) + \varphi(t)) + o(\epsilon^3). \end{aligned} \tag{17b}$$

An attempt is now made to provide information on the accuracy of AP method by using comparison with direct numerical integration of (2a)–(2b) and it is demonstrated that satisfactory results are obtained for many varied situations. Some solutions of (2a)–(2b) have been obtained numerically using a fifth order Runge–Kutta–Fehlberg integration scheme. It is possible to estimate a reasonable error bound directly from the perturbation method, as terms of order ϵ^3 have been neglected and then the difference between numerical and analytical solutions must be of the same magnitude order.

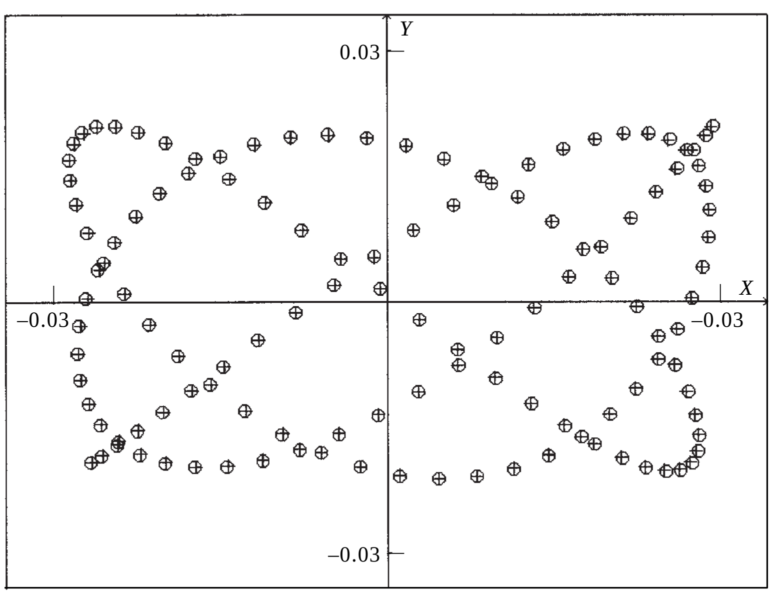

Figure 1. Representation in the X–Y space of an orbit with the set of parameters defined in equation (18). ○ numerical solution; + approximate solution.

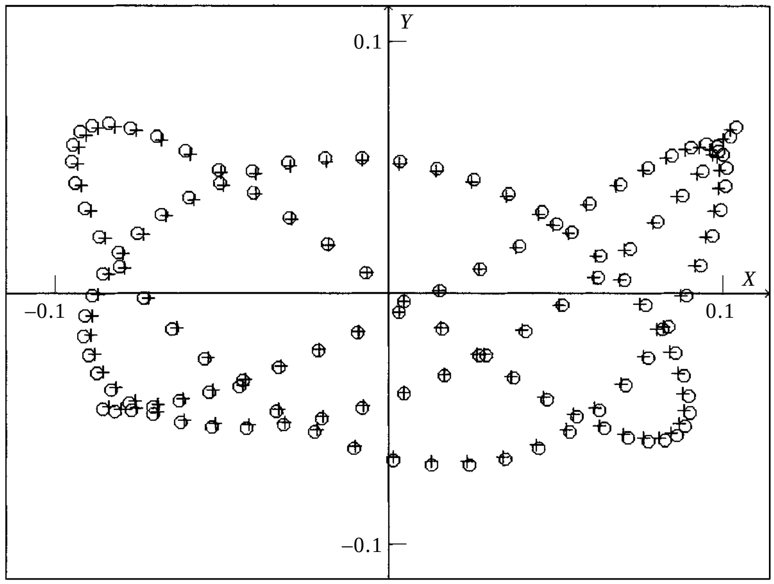

Figure 2. X–Y orbit with the same set of parameters of Figure 1 but with $a_1 = -0{\cdot}02$, $b_1 = 0{\cdot}01$.

Figure 1 shows a representation in the X–Y space of an orbit with the following set of parameters:

$$\omega_1 = 1, \qquad \omega_2 = \sqrt{2}, \tag{18a}$$

$$a_1 = -0{\cdot}002, \qquad a_4 = -1, \qquad a_{13} = -2, \qquad a_{16} = 1{\cdot}5,$$

$$a_2 = a_3 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{11} = 1 \tag{18b}$$

$$a_{12} = a_{14} = a_{15} = a_{17} = a_{18} = a_{19} = 1,$$

$$b_1 = 0{\cdot}001, \qquad b_2 = b_3 = b_5 = -2, \qquad b_4 = b_{16} = 1{\cdot}5,$$

$$b_6 = b_7 = b_8 = b_{10} = -1, \tag{18c}$$

$$b_9 = b_{11} = b_{12} = b_{13} = b_{14} = b_{15} = b_{17} = b_{18} = b_{19} = 1.$$

Note that $\varepsilon \approx 0{\cdot}04$, because ε^2 is the magnitude order of linear dissipative terms. The point $P_4 \equiv (0{\cdot}0132; 0{\cdot}0098)$ is asymptotically stable and the initial conditions have been chosen on the invariant torus corresponding to P_4. Circles correspond to the numerical solution and crosses to the approximate solution (17a)–(17b). The numerical solution and the approximate solution overlap almost entirely and the maximum different between the two solutions is 0·00080 and the mean difference is 0·00025, i.e. of order ε^3. The agreement of the results appears to be excellent.

In Figure 2 the same set of parameters as for Figure 1 are considered but with $a_1 = -0{\cdot}02$, $b_1 = 0{\cdot}01$, $\varepsilon \approx 0{\cdot}1$ (an increase of an order of magnitude). The point $P_4 \equiv (0{\cdot}042; 0{\cdot}031)$ is always asymptotically stable and the maximum difference between the numerical and the approximate solution is 0·0085 and the mean difference is 0·0025, i.e. of order ε^3 as expected.

The bifurcation diagram can be easily determined, taking into account that the involved parameters are only α_2, β_2, γ_1. The behaviour of the system (2a)–(2b) is then intelligible in terms of a very limited number of parameters.

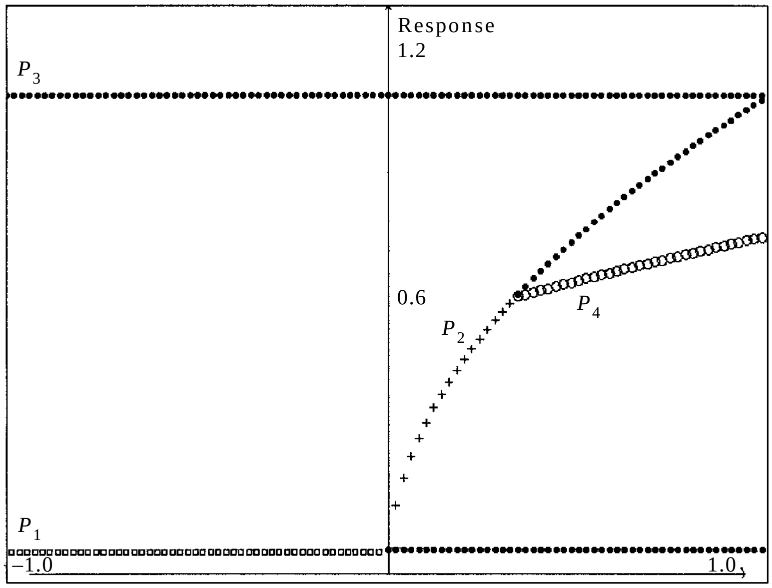

Figure 3. Response-parameter (α_2) diagram: squares stand for P_1, crosses for P_2 and circles for P_4. Dots represent unstable solutions. $\alpha_1 = -1$, $\beta_1 = 1$, $\gamma_1 = 3$, $\beta_2 = -3$, $\gamma_2 = -1$.

The AP method enables one to reduce the number of the effective parameters and to calculate the opportune combinations of the starting parameters suitable to produce the universal parameters α_2, β_2, γ_1.

In Figures 3, 4 and 5 we show a parameter (α_2)-response diagram: the response is represented by the amplitude of the relative equilibrium point. Squares stand for P_1, crosses for P_2, rectangles for P_3 and circles for P_4, if they are asymptotically stable,

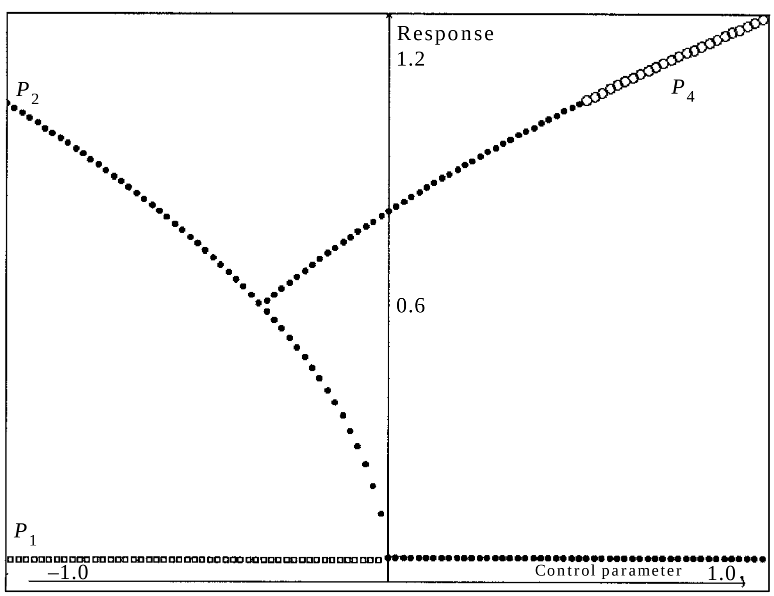

Figure 4. Response-parameter (α_2) diagram: squares stand for P_1 and circles for P_4. Dots represent unstable solutions. $\alpha_1 = -1$, $\beta_1 = -1$, $\gamma_1 = 3$, $\beta_2 = -2$, $\gamma_2 = 1$.

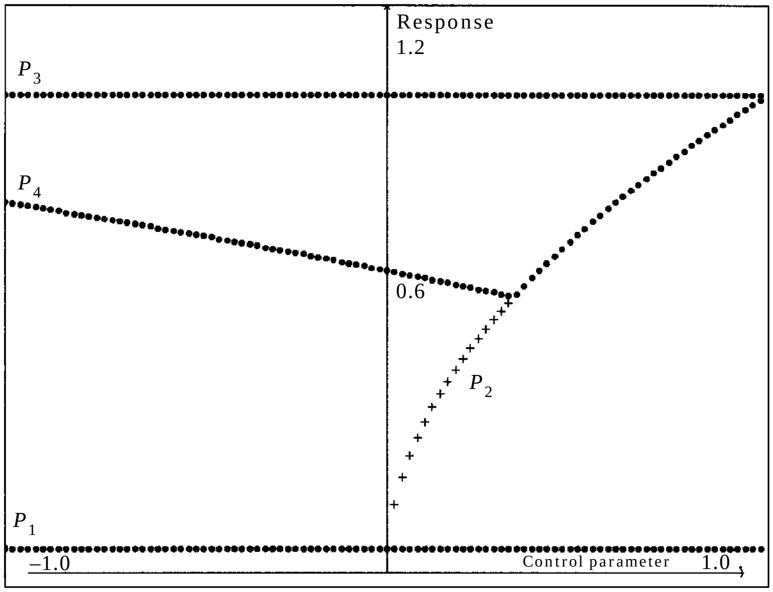

Figure 5. Response-parameter (α_2) diagram: crosses stand for P_2. Dots represent unstable solutions. $\alpha_1 = -1$, $\beta_1 = 1$, $\gamma_1 = 3$, $\beta_2 = 3$, $\gamma_2 = -1$.

otherwise they are represented by simple dots. Our results are in qualitatively agreement with those of the center manifold reduction and the normal form theory [4, section 7.5].

4. TWO-ROD SYSTEM UNDER AERODYNAMIC EXCITATION

In this section, the perturbation method described above is applied to a mechanical system composed of two rods under aerodynamic excitation [22]. Consider a structure composed of two vertical rigid rods of length L, constrained by two visco-elastic hinges of torsion rigidity K_H and damping coefficient $C_H > 0$. The structure is loaded by a fluid flow of uniform velocity V in a direction orthogonal to the plane of motion. The rods are joined at their ends by a visco-elastic device, that either puts or dissipates energy into the system, whose rigidity is K_D and damping coefficient C_D. An aerodynamic force, depending on V and on the shape of the cross-section, arises in the plane of motion. As a consequence in particular conditions one observes a galloping instability (Hopf bifurcations). By applying the quasi-static theory for aerodynamic forces and expanding non-linearities up to the third order, gives the following non-dimensional equations of motion [22]

$$\ddot{X} + (f_1 - fv)\dot{X} + X = 2hXY^2 + 4fAXY\dot{Y} + C_2(\dot{X}^2 + \dot{Y}^2) + \frac{C_3}{v}(\dot{X}^3 + 3\dot{X}\dot{Y}^2) \quad (19a)$$

$$\ddot{Y} + (f_2 - fv)\dot{Y} + \omega_0^2 Y = 2hXY^2 + \tfrac{4}{3}hY^3 + 4fA(X^2\dot{Y} + XY\dot{X} + Y^2\dot{Y}) + C_2\dot{X}\dot{Y} + \frac{C_3}{v}(\dot{Y}^3 + 3\dot{X}^2\dot{Y}) \quad (19b)$$

where

$$v = \frac{\rho b}{m\omega_x}V, \qquad \omega_x^2 = \frac{3K_H}{mL^3}, \qquad h = \frac{2K_D L^2}{K_H}, \qquad \omega_0^2 = 1 + h, \quad (20a)$$

$$f_1 = \frac{3C_H}{mL^3\omega_x}, \qquad f_2 = f_1 + fA, \qquad f = \frac{|C_D + C_L'|}{2}, \qquad A = \frac{3C_D}{mL\omega_x f}, \quad (20b)$$

$$C_2 = \frac{3}{16}\left(\frac{\rho b L}{m}\right)(C_L'' + C_L + 2C_D'), \tag{20c}$$

$$C_3 = -\frac{1}{20}\left(\frac{\rho b L}{m}\right)^2(C_L''' + C_L' + +3C_D'' + 3C_D) \tag{20d}$$

where the dot denotes differentiation with respect to non-dimensional time $\tilde{t} = \omega_x t$, $\omega_0 = \omega_y/\omega_x$ is the ratio between the two undamped frequencies, assumed to be incommensurable, C_D and C_L are the drag and lift non-dimensional coefficients, respectively; C_D', C_D'', C_L', C_L'' and C_L''' are their derivatives with respect to the attack angle, ρ is the air density, b is an appropriate characteristic length of the cross-section of the rods, m is the mass per unit length of rods, f_1 and f_2 are the modal structural dampings, f is the aerodynamic modal damping, A is the non-dimensional damping of the visco-elastic device, assumed as the first control parameter and v is the non-dimensional wind velocity. In equations (19a, b), modal co-ordinates have been used in order to uncouple the linear part, namely

$$X = \tfrac{1}{2}(q_1 + q_2), \qquad Y = \tfrac{1}{2}(q_1 - q_2) \tag{21}$$

where Lagrangian co-ordinates q_1 and q_2 represent the angles formed by rods with the vertical axis. X is then the amplitude of the antisymmetric ($q_1 = 1$, $q_2 = 1$) mode and Y the amplitude of the symmetric ($q_1 = -1$, $q_2 = 1$) mode. Equations (19a, b) have been studied by the multiple scales method in reference [8].

If the coefficients of the velocities $\dot{X}$ and $\dot{Y}$ in equations (19a, b) vanish, the trivial equilibrium position $X = Y = 0$ loses its stability through a Hopf bifurcation. An antisymmetrical and a symmetrical galloping mode are produced by the instability which happens for two critical wind velocities $v_1 = f_1/f$ and $v_2 = f_1/f + A$, respectively. By posing $B = v - v_1$, and assuming B as the second control parameter, it easily be concluded that

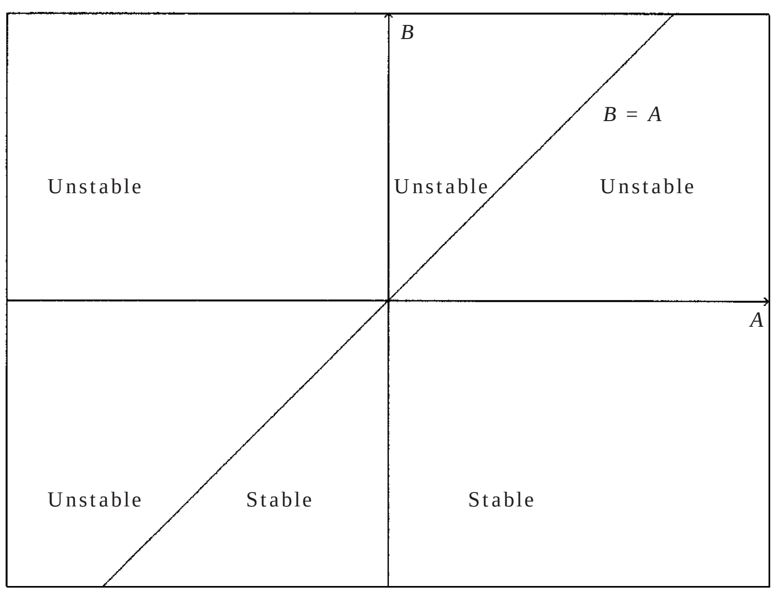

Figure 6. Stability of the trivial path in the parameter space (A, B).

the trivial path is stable for $B < 0$ and $A > B$. Two successive Hopf bifurcations associated with the symmetric and the antisymmetric modes happen for negative damping A and increasing B; for positive damping A the two bifurcations occur in the reverse order (Figure 6). The two boundary stability curves are determined by the equations $B = 0$ and $A = B$. Equations (2a, b) are identical to equations (19a, b), if one sets

$$\omega_1^2 = 1, \qquad \omega_2^2 = \omega_0^2, \qquad a_1 = fv - f_1, \qquad b_1 = fv - f_2,$$
$$a_{16} = b_{16} = 2h, \qquad a_{20} = b_{17} = b_{20} = b_{13} = 4fA, \tag{22}$$
$$a_4 = a_9 = b_{11} = C_2, \qquad a_{19} = b_{15} = 3b_{19} = 3a_{15} = \frac{C_3}{v}, \qquad b_{12} = \tfrac{4}{3}h,$$

and all the other coefficients are zero. The system model (15a)–(15d) yields

$$\frac{d\rho}{d\tau} = \frac{1}{2} fB\rho + \frac{C_3}{v_1}\omega_0^2\rho\chi^2 + \frac{3}{2}\frac{C_3}{v_1}\rho^3 \tag{23a}$$

$$\frac{d\chi}{d\tau} = \frac{1}{2} f(B - A)\chi + \frac{C_3}{v_1}\rho^2\chi + \frac{3}{2}\frac{C_3}{v_1}\omega_0^2\chi^3 \tag{23b}$$

and

$$\frac{d\vartheta}{d\tau} = \tfrac{2}{3}C_2^2\rho^2 + 2\left(\frac{\omega_0^2 C_2^2(2\omega_0^2 - 1)}{4\omega_0^2 - 1} + 2h\right)\chi^2 \tag{23c}$$

$$\frac{d\varphi}{d\tau} = \left(\frac{\omega_0 C_2^2}{4\omega_0^2 - 1} + 2\frac{h}{\omega_0}\right)\rho^2 + \left(\frac{1}{3}\omega_0 C_2^2 + 2\frac{h}{\omega_0}\right)\chi^2. \tag{23d}$$

The amplitude equations (23a, b) are uncoupled from the phase equations (23c, d). It is now necessary to determine the steady-state solutions of the dynamical system and to perform the stability analysis.

Equations (23a, b) admit the trivial solution $(\rho_1, \chi_1) = (0, 0)$, but three non trivial steady-state solutions with one or two non vanishing components are possible:

$$\rho_2 = \sqrt{-\frac{v_1 fB}{3C_3}}, \qquad \chi_2 = 0 \tag{24}$$

$$\chi_3 = \sqrt{\frac{v_1 f(A - B)}{3C_3\omega_0^2}}, \qquad \rho_3 = 0 \tag{25}$$

$$\rho_4 = \sqrt{-\frac{2v_1 f}{C_3}\left(\frac{B}{2} + A\right)}, \qquad \chi_4 = \sqrt{-\frac{v_1 f(B - 3A)}{5C_3\omega_0^2}}, \tag{26}$$

while the corresponding ϑ_j and φ_j, $j = 2, \ldots, 4$ are obtained by direct substitution of (24)–(26) in (23c, d).

Both solutions (24) and (25) correspond to periodic responses of the original system (19a, b), while the resultant motion of solution (26) is quasi-periodic, since the frequencies of the two interacting modes are incommensurable. First order approximations of the original system (19a, b) are given by (17a, b). Solutions (24) and (25) exist only for certain ranges of the control parameters, depending on the sign of C_3. For example, if $C_3 > 0$

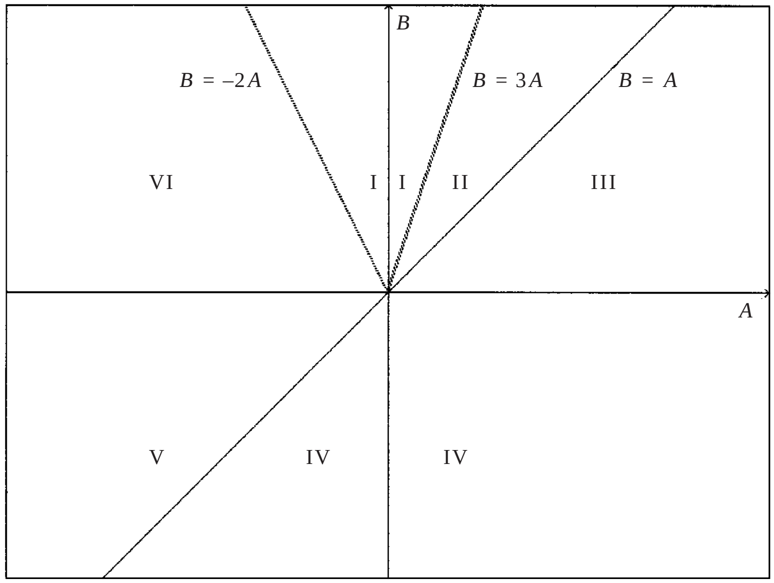

Figure 7. Response-parameter (B) diagram: bifurcated steady-state amplitudes vs B for (a) $A > 0$ and (b) $A < 0$ for $C_3 < 0$.

solution (24) exists for $B < 0$ and solution (25) for $A > B$, while the domain of definition of solution (26) is $B + 2A > 0$ and $B - 3A > 0$, since ρ and χ are real and positive.

Now consider the stability and bifurcation analysis.

Let (ρ_j, χ_j), $j = 1, \ldots, 4$, be a steady-state solution to equation (23a, b). The stability of (ρ_j, χ_j) depends on the Jacobian matrix $\mathbf{J}$

$$\mathbf{J} = \begin{bmatrix} \dfrac{fB}{2} + \dfrac{C_3}{v_1}(\omega_0^2\chi_j^2 + \tfrac{9}{2}\rho_j^2) & \dfrac{2C_3\omega_0^2}{v_1}\rho_j\chi_j \\ \dfrac{2C_3\rho_j\chi_j}{v_1} & \dfrac{f(B-A)}{2} + \dfrac{C_3}{v_1}(\tfrac{9}{2}\omega_0^2\chi_j^2 + \rho_j^2) \end{bmatrix}. \tag{27}$$

A geometric representation of the bifurcation analysis has been drawn in Figure 7, obtained for $C_3 < 0$, in which the stability of the various equilibrium points is determined for different regions of the control parameters plane. Six different regions are distinguished.

(I) the quasi-periodic solution (26) is stable, while the other solutions exist but are unstable;

(II) the periodic antisymmetric solution (24) is stable, while the solution (25) and the trivial path exist but are unstable;

(III) the periodic antisymmetric solution (24) is stable, while only the trivial path exists but is unstable;

(IV) only the trivial path exists and is stable;

(V) the periodic symmetric solution (25) is stable, while only the trivial path exists but is unstable;

(VI) the periodic symmetric solution (25) is stable, while the solution (25) and the trivial path exist but are unstable.

Considering $A < 0$ and increasing B, one concludes that the trivial equilibrium position loses its stability after a pitchfork bifurcation and the stable symmetric solution (25) arises. For increasing values of B, a second static bifurcation is observed and the unstable

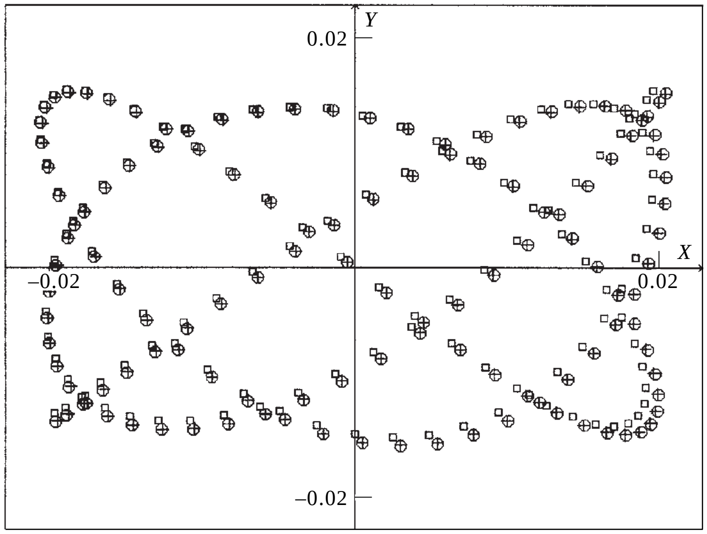

Figure 8. Representation in the X–Y space of an orbit of the system (19a, b) with the following set of parameters: $\omega_1 = 1$, $\omega_2 = \sqrt{2}$, $A = 0{\cdot}25$, $B = 1$, $f = 0{\cdot}001$, $v = C_2 = C_3 = 1$, $h = -1$. ○ numerical solution, + AP method, □ multiple scales method.

antisymmetric mode (26) appears. Finally, if B is further increased, the antisymmetric periodic solution (26) bifurcates in the quasi-periodic stable solution (27). An analogous discussion can be applied for $A > 0$. The trivial equilibrium position loses its stability after a pitchfork bifurcation and the stable antisymmetric solution (26) arises. For increasing values of B, one observes a second static bifurcation and the unstable symmetric mode (25) appears. Finally, if B is further increased, the symmetric periodic solution (25) bifurcates into the quasi-periodic stable solution (27). In the particular case $A = 0$, all the bifurcation points coalesce and a unique stable steady-state quasi-periodic motion exists, directly bifurcating from the trivial path.

Our results are in qualitative agreement with the analysis of reference [8], but the first order approximations (17a, b) are slightly different, due to the different perturbation methods used. In Figure 8, the results obtained by the AP method are compared with those obtained in reference [8]. For the chosen set of parameters, the AP method approximate solution is slightly better of the multiple scales method solution.

5. CONCLUSION

A perturbation analysis for a system of two non-linearly coupled oscillators resulting in a set of ordinary differential equations that depends on five essential parameters only has been presented. The analytical results are then used for a bifurcation analysis. Amplitude equations have been derived which describe the solutions of non-linear systems as superpositions of harmonic terms, the amplitude of which is modulated by the non-linear terms. The solution is written as a Fourier expansion in which the coefficients are power series of a small parameter and vary slowly in time. The lowest order terms correspond to the harmonic solutions of the linear problem. Dynamic equations for the amplitudes of these harmonic terms are then derived by substituting the expression of the solution into the original equation and projecting onto each Fourier component. This technique has been applied to a very general system of non-linear differential equations. The model equations in polar form (15) are used to locate invariant tori depending on three arbitrary parameters. Finally, numerical evidence has been presented for the accuracy of the amplitude equations as approximations to the original systems. The analytical results are then compared with numerical solutions.

Finally, the results have been applied to the analysis of the post-critical behavior of a two-rod system under aerodynamic excitation.

Some possible extensions of the technique described in this paper are:

(i) application of the reduction method beyond its leading order;

(ii) study of the special case

$$\beta_1\gamma_1 = \beta_2\gamma_2, \qquad \alpha_2 = -\alpha_1 \tag{28}$$

that implies the existence of a first integral

$$F(\chi, \rho) = (\rho\chi)^A(1 + B\rho^2 + C\chi^2), \tag{29a}$$

$$A = \frac{-2\beta_1}{\beta_1 + \beta_2}, \qquad B = \frac{\beta_1}{\alpha_1}, \qquad C = -\frac{\gamma_2}{\alpha_1}, \tag{29b}$$

i.e. of a function constant along the solution curves of (15a)–(15b);

(iii) study of the special case $\beta_2 = \gamma_1$: in this case system (15a)–(15b) can be written as

$$\dot{X} = \nabla U(X), \qquad X = (\rho, \chi) \tag{30a}$$

$$U(\rho, \chi) = \alpha_1\frac{\rho^2}{2} + \alpha_2\frac{\chi^2}{2} + \beta_1\frac{\rho^4}{4} + \gamma_2\frac{\chi^4}{4} + \gamma_1\frac{\chi^2\rho^2}{2}; \tag{30b}$$

(iv) generalization to three or more coupled oscillators leading to similarly universal set of equations;

(v) parametric vibrations: single- and multi-frequency excitations of the non-linear coupled oscillators (4a)–(4b).

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